$\boxed{12}$

1 a) $\phi_j^{n+1} = \left(\frac{\eta}{2}+\frac{d}{2}\right)\phi_{j-1}^{n} + (1-d)\phi_j^{n} + \left(-\frac{\eta}{2}+\frac{d}{2}\right)\phi_{j+1}^{n}$

1

grid $[0 \;\; 0.5 \;\; 1]$ $\Delta x = 0.5$ $\Delta t = 0.1$ $u = 4$ $k = 0.05$

$\eta = 0.8$ $d = 0.04$

$\phi_j^{n+1} = 0.42\,\phi_{j-1}^{n} + 0.96\,\phi_j^{n} + (-0.38)\,\phi_{j+1}^{n}$

		$x_1 = 0$	$x_2 = 0.5$	$x_3 = 1$	
	$n = 0$:	0	0.5	1	initial cond
1	$n = 1$:	0	0.1	0.1	use $\phi_3 = \phi_2$
1	$n = 2$:	0	0.058	0.058	1

Because of boundary treatment $\phi_3 = \phi_2 \Rightarrow$

1 $\phi_2^{n+1} = \underbrace{0.42\,\phi_1^{n}}_{=0} + \underbrace{0.058}_{>0}\,\phi_2^{n}$ positive coef $\Rightarrow$ stable

2 max Δt: $\frac{\eta}{2}+\frac{d}{2} > 0$ o.k. $(1-d)+\left(-\frac{\eta}{2}+\frac{d}{2}\right) = 1 - \frac{d}{2} - \frac{\eta}{2} > 0$ $\delta t < \frac{2\Delta x^2}{2k+u\Delta x} = 0.2381$

2 $t \to \infty$ $0.42\,\phi_{j-1}^{n} - 0.04\,\phi_j - 0.38\,\phi_{j+1} = 0$

$\Rightarrow 0.42\,\phi_{j-1}^{n} = 0.42\,\phi_j^{n}$ positive coef $\Rightarrow$ no spatial wiggle

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Different treatment of bou c at $x=1$ $\Rightarrow$ different values

total: 14

General analysis : 2 $P = 40 > 2 \rightarrow$ wiggles

1 $\left.\begin{array}{l} \eta^2 < d < 1 \\ P < 2 \wedge d < 1 \end{array}\right\}$ not satisfied $\Rightarrow$ unstable

2 max Δt: $d < 1 \Rightarrow \Delta t < \frac{\Delta x^2}{2k} = 2.5$

$\eta^2 < d \Rightarrow \Delta t < \frac{2k}{u^2} = \cancel{}$ 0.00625 $\leftarrow$ limit

1st order upwind : $d \rightarrow d+\eta$

1 $\phi_j^{n+1} = \left(\eta + \frac{d}{2}\right)\phi_{j-1}^{n} + (1-d-\eta)\phi_j^{n} + \left(\frac{d}{2}\right)\phi_{j+1}^{n}$

$\quad 0.82 \qquad 0.16 \qquad 0.02$

1 if $\phi_3 = \phi_2$ $\quad \phi_2^{n+1} = 0.82\,\phi_1^{n} + 0.18\,\phi_2^{n}$ positive coef

1 no Peclet wiggles $\quad \simeq 0 \quad \gtrsim 0 \quad \Rightarrow$ stable

General analysis: less spatial accuracy (truncation error)

2 but no spatial wiggles anymore $\rightarrow$ better quality

1 stability: $\eta < 1-d \quad \delta t < \frac{\Delta x^2}{2k + u\Delta x} = 0.11905$

$\boxed{6}$

b) 2 $\left(-\frac{\eta}{2} - \frac{d}{2}\right)\phi_{j-1}^{n+1} + (d+1)\phi_j^{n+1} + \left(\frac{\eta}{2} - \frac{d}{2}\right)\phi_{j+1}^{n+1} = \phi_j^{n}$

$-0.42 \qquad 1.04 \qquad 0.38$

1.42 ($\phi_3 = \phi_2$ treatment)

$n=0$	0	0.5	1
1 $n=1$	0	$\frac{0.5}{1.42} = 0.35211$	0.35211
1 $n=2$	0	$\frac{0.35211}{1.42} = 0.24797$	0.24797

1 implicit $\Rightarrow$ stable

1 same presence/absence spatial wiggles as explicit

$\boxed{8}$

2 $\quad \frac{u}{\Delta x}\left(\frac{3}{2}\phi_i - 2\phi_{i-1} + \frac{1}{2}\phi_{i-2}\right) = \frac{k}{\Delta x^2}\left(\phi_{i+1} - 2\phi_i + \phi_{i-1}\right)$

1 $\quad \Rightarrow \frac{-k}{\Delta x^2}\phi_{i+1} + \underbrace{\left(\frac{2k}{\Delta x^2} + \frac{3}{2}\frac{u}{\Delta x}\right)}_{>0}\phi_i + \left(\frac{-k}{\Delta x^2} - \frac{2u}{\Delta x}\right)\phi_{i-1} + \frac{1}{2}\frac{u}{\Delta x}\phi_{i-2} = 0$

2 $\quad \frac{-k}{\Delta x^2} < 0$ o.k.

alternating signs before coeff.

2 $\quad \frac{-k}{\Delta x^2} - \frac{2u}{\Delta x} < 0$ o.k.

$- \quad + \quad - \quad +$

2 $\quad \frac{1}{2}\frac{u}{\Delta x} > 0$ o.k.

1 always wiggle free for every $r > 0$

Theorem reader (exercise) : $\lambda \geq \max\left(\frac{1}{2} - \frac{k}{u\Delta x}, 0\right) \Rightarrow$ wiggle free

$B3 \Leftrightarrow \lambda = 1/2 \qquad \frac{1}{2} \geq \max\left(\frac{1}{2} - \frac{k}{u\Delta x}, 0\right)$

$\Leftrightarrow$ o.k.

$\phi_i = r^i \Rightarrow \frac{-k}{\Delta x^2} r^3 + \left(\frac{2k}{\Delta x^2} + \frac{3}{2}\frac{u}{\Delta x}\right) r^2 + \left(\frac{-k}{\Delta x^2} - \frac{2u}{\Delta x}\right) r + \frac{1}{2}\frac{u}{\Delta x} = 0$

1 $\qquad$ 1

$r = 1$ is sol $\Rightarrow (r-1)\left(\frac{-k}{\Delta x^2} r^2 + \left(\frac{3}{2}\frac{u}{\Delta x} + \frac{k}{\Delta x^2}\right) r - \frac{u}{2\Delta x}\right) = 0$

1

$\hookrightarrow$ solutions $r > 0$ check

3

always $r > 0 \Rightarrow$ no wiggles !

3 $\boxed{12}$ a) grid $[0, 1-2k, 1]$ $h_- = 1-2k$ $h_+ = 2k$ $h_+ + h_- = 1$

1 $$\phi_{xx} = \frac{h_- \phi_+ - (h_+ + h_-)\phi_0 + h_+ \phi_-}{\frac{1}{2} h_+ h_- (h_+ + h_-)}$$

$$= \frac{(1-2k)\phi_+ - \phi_0 + (2k)\phi_-}{\frac{1}{2}(2k)(1-2k)}$$

$$\Rightarrow k\phi_{xx} = \frac{k(1-2k).1 - k\phi_0}{k(1-2k)}$$

1 $$\phi_{x_A} = \frac{h_+}{h_+ + h_-}\frac{\phi_+ - \phi_0}{h_+} + \frac{h_-}{h_+ + h_-}\frac{\phi_0 - \phi_-}{h_-} = \phi_+ - \phi_- = 1$$

1 $$\phi_{x_B} = \frac{h_-}{h_+ + h_-}\frac{\phi_+ - \phi_0}{h_+} + \frac{h_+}{h_+ + h_-}\frac{\phi_0 - \phi_-}{h_-} = \frac{(1-2k)^2}{2k(1-2k)}(1-\phi_0) + \frac{(2k)^2}{2k(1-2k)}(\phi_0 - 0)$$

unsteady conv. diff : A : $1 = \frac{k(1-2k)1 - k\phi_0}{k(1-2k)} \Rightarrow \phi_0 = 0$

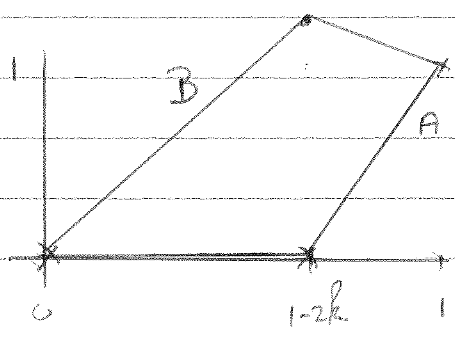
B : $$\frac{(1-2k)^2(1-\phi_0) + (2k)^2\phi_0}{2k(1-2k)} = \frac{2k(1-2k) - 2k\phi_0}{2k(1-2k)}$$

2

$$\Rightarrow (1-2k)^2 - 2k(1-2k) = \left((1-2k)^2 - (2k)^2 - 2k\right)\phi_0$$
$$\Rightarrow 1 - 6k + 8k^2 = (1 - 6k + 4k^2 - 4k^2)\phi_0$$

$$\phi_0 = \frac{1 - 6k + 8k^2}{1-6k} = 1 + \frac{8k^2}{1-6k}$$

2



k small $\frac{8k^2}{1-6k} \hat{>} 0$

exact sol: $\phi(x) = \phi_0 + (\phi_1 - \phi_0)\dfrac{1 - e^{Pe\frac{x}{L}}}{1 - e^{Pe}}$ $\quad Pe = \dfrac{u \cdot L}{k} = \dfrac{1}{k}$

1 $\quad = \dfrac{1 - e^{x/k}}{1 - e^{1/k}}$

1 $\phi(1-2k) = \dfrac{1 - e^{(1-2k)/k}}{1 - e^{1/k}} = \dfrac{1 - e^{1/k} \cdot e^{-2}}{1 - e^{1/k}}$

1 meth A better, no overshoots (wiggles)
meth B wiggles, strongly depends on value of k.

$\boxed{6}$

b) 1 diffusion both methods symmetric
convection meth A skew symmetric
1 " B depends on parameters

! $\begin{pmatrix} \ddots & & & \\ L & D & R & \\ & L & D & R \\ & & & \ddots \end{pmatrix}$ $\quad L_i = R_i$ for symmetry
$L_i = -R_i$ skew symmetry

3 eigenvalues meth A in right half plane: $Re(\lambda) > 0$
$\Rightarrow$ matrix non singular
" " B may be elsewhere $\rightarrow$ chance for singularities (and large errors)

4 $\boxed{6}$

a) $t = t + \Delta t$

1 set up Poisson eqn, include bound. for Pressure derived from bound. for velocity

2 solve Poisson $\Delta p^{n+1} = \frac{\nabla . u^n}{\Delta t} + \nabla . R^n$ ← conv. diff

2 update velocity $u^{n+1} = u^n + \Delta t R^n - \Delta t \nabla p^{n+1}$

1 set boundary velocities

next time step

$\boxed{4}$

b) 1 prevents error accumulation regarding mass conservation

2
$$\frac{1}{\Delta t} (\underbrace{\nabla . u^{n+1}}_{= 0} - \nabla . u^n) + \Delta p^{n+1} = \nabla . R^n$$

$$\Delta p^{n+1} = \frac{1}{\Delta t} \underbrace{\nabla . u^n}_{\approx 0} + \nabla . R^n$$

numerically $\Delta p^{n+1} = \nabla R^n + \varepsilon^{n+1}$

1 $\nabla . u^{n+1} = \nabla . u^n + \Delta t \varepsilon^{n+1}$

5 $\boxed{3}$

a) $(10^8)^{9/4}$ points

10^5 flops per point, each time step

10^4 time steps

$\Rightarrow (10^8)^{9/4} \cdot 10^5 \cdot 10^4 = 10^{27}$ flops

machine 10^{18} flops/sec $\Rightarrow 10^9$ sec

($\hat{=}$ 32 years)

3 b) 1 higher order discretisation (less points)

1 faster matrix algorithms

1 faster computer